Motivating and Engaging Students in Active Learning of Mathematics: Mathematical Constants and Applications in Undergraduate Studies

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Mathematics presents many challenges to both students and mathematics educators. In this paper, I'd share with my colleagues about my exploration in teaching and learning mathematics along with my students. Specifically, I will present three most commonly used mathematical constants and their applications in undergraduate studies. The constants are the circumference ratio π, the golden ratio φ, and the Euler's number e. I will demonstrate how I use these concrete mathematical materials to motivate and engage students with diverse backgrounds in an active learning environment. For each constant, I will present its brief history along with its applications in science, business and social science.

This paper is a recollection of my presentations at the annual AURCO conferences in the past three years. I have been teaching undergraduate mathematics over a decade. As a longtime mathematics educator, I have the privilege to work with many college students with diverse backgrounds. There are enthusiastic students who are really interested in mathematics and there are students who take mathematics simply as a course requirement for graduation. There are young students in their teens and adults in their middle career, or even after retirement. There are students with diverse culture background from different regions, states and countries. There is also a significant number of students who are first generation college students.

Each student is unique in his/her way of learning, preparedness and motive. As an educator, besides being an expert in the subject I teach, I am mindful to be flexible to adapt instruction materials and to present the mathematical content in accessible ways for individuals. The breadth and depth of mathematical constants along with their applications allows me to choose appropriate topics for students with diverse backgrounds.

Teaching mathematics is a complex practice demanding creative and critical thinking from both students and educators. When I teach mathematics in the classroom, I consider myself a role model of learning. In planning and conducting instruction, I always make sure that students understand that abstract mathematical ideas and concepts are utilities to solve real world problems. Mathematical constants are building blocks in many areas of studies in science, business and social science. While occurring at all levels in many majors of undergraduate studies, mathematical constants involve with various fundamental applications in human life. Here are some examples in which mathematical constants play important roles. The circumference ratio is found in the applications of recording mileages, designing pendulums and tracking GPS signals. The golden ratio is critical in architecture design, weather forecast and economic growth. The Euler's number is the base of the exponential function, of which the growth rate is proportional to itself. Many natural and social phenomena rise from the fact that the growth rate is proportional to its size. Accordingly, the Euler's number is uniquely critical to measure the exponential growth, radioactive decaying, and compound interest.

Teaching and learning mathematics are evolving with the advance of technology. A common misperception is that mathematics relies on technology. Under the false assumption, there are students who rely heavily on using technology, such as a calculator, when taking mathematical classes. I encourage my students to use technology as a tool in learning mathematics. Furthermore, I challenge them to find how technology handles mathematical tasks. For instance, series representations of the mathematical constants discussed in this paper are essential in implementing transcendental functions in computer software. Mathematics is the foundation of technology. By studying mathematics well, we will make technology better.

Mathematics presents many challenges to both students and educators. In this paper, I share with my colleagues my exploration in teaching and learning mathematics along with my students. Specifically, I will present three most commonly used mathematical constants and their applications in undergraduate studies. The constants are the circumference ratio π, the golden ratio φ, and the Euler's number e. I will demonstrate how I use these concrete mathematical materials to motivate and engage students with diverse backgrounds in an active learning environment. For each constant, I will present its brief history along with applications in science, business and social science.

1. The Circumference Ratio π

The ratio of the circumference of a circle to its diameter as a constant has been known from the very early stage of human history. The ancient Babylonians calculated area of a circle by taking three times the square of its radius, which provided the ratio a value of 3. Such a value was recorded on a Babylonian tablet (ca. 1900–1680 BC). Later, the Egyptians were able to improve the approximation to a value of 3.1605 as shown in the Egyptian Rhind Papyrus (ca.1650 BC).

It is believed that the approximations used by the Babylonians and the Egyptians were found by measurement. The first recorded calculation was done by Archimedes of Syracuse (287 – 212 BC). Archimedes was generally considered the greatest mathematician in the classical era. He calculated the circumference ratio by using two polygons, one inscribed in the circle, and the other within which the circle was inscribed. The area of the circle is thus bounded by the areas of the polygons. Archimedes showed that the ration is between $3\frac{1}{7}$ and $3\frac{10}{71}$. More importantly, the idea used by Archimedes laid foundation to the development of calculus.

After 750 years or so, a talented Chinese mathematician, Chongzhi Zu (429 – 501), calculated the value of the circumference ratio to be $\frac{335}{113}$ by using the polygon approximation. Because of the great distance and limited communication from west to east, Zu would not have known Archimedes' work. Zu's work was independent and remarkable.

Partly due to Archimedes' contribution, mathematician started to use the Greek letter π to denote the ratio of the circumference of a circle to its diameter in 1700s. The symbol has been popularly used worldwide since the adoption by another mathematics giant, the Swiss mathematician Euler.

Despite several thousand years of efforts, we still don't know π exactly. It is an irrational number, which means it has infinite number of decimal digits without any pattern. It is approximately equal to 3.1415926535897932. With the advance of computing power from modern computers, researchers can generate billions of digits of the circumference ratio π of a circle to its diameter.

Thanks to its wide existence and abundant applications, the circumference ratio π is an active ingredient to engage student centered learning. When I teach subjects related to it, I encourage my students to think creatively and critically. As a creative thinking component, I challenge them to explore new methods to evaluate it. Here is one example I used in my calculus class. We know

$$\frac{d}{dx} tan^{-1}x = \frac{1}{1+x^2},$$

which is equivalent to the integral form,

$$\int \frac{1}{1+x^2} dx = tan^{-1}x + C.$$

From power series, we have

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \cdots.$$

Then, we get

$$tan^{-1}x + C = \int \frac{1}{1+x^2} dx$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 \cdots$$

by integrating the series term by term. When $x = 0$, we can find the constant $C = 0$. Thus,

$$tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 \cdots.$$

When $x = 1$, we obtain

$$\frac{\pi}{4} = tan^{-1}1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots.$$

Not surprisingly, we can find π,

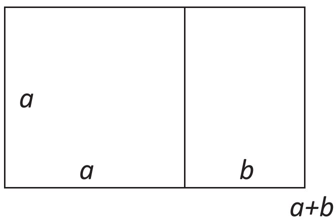
$$\pi = 4 \times tan^{-1}1 = 4 \times (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots).$$

The circumference ratio π is one of the most commonly known fundamental mathematical constants. Its applications range from all areas of science as well as business and social science. As a critical thinking component, I ask my students to find its new applications in modern life, especially from their own experience. I am surprised to see a wide variety of applications they come up with, most of which are closely related to our daily life. For example, by counting the number of rotations of wheels, we can track the mileage of a vehicle may have traveled. Wireless signals are electronic waves, which are measured in terms of π. Consequently, it plays an important role in the implementation of the global positioning system (GPS).

2. The Golden Ratio φ

Mathematics presents in every aspect of human life and nature. Just like the circumference ratio π, the golden ratio φ has extensive presence and application in many aspects of human life and nature, such as art, music, business as well as biology. It is also known by a few other names, such as the golden mean, the divine proportion and the golden cut. Historically, Euclid provided the first written definition of the golden ratio in his masterpiece Elements.

A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the lesser.



Mathematically, it means

$$\frac{a+b}{a} = \frac{a}{b}$$

as shown in the figure. It is believed that rectangles created such that the ratio of length to width equal to the golden ratio are most beautiful from the aesthetic point of view. Indeed, the golden ratio is found in many historical sites, including the Great Pyramid of Giza in Egypt and the Parthenon in Greece. Leonardo Da Vinci called the golden ratio the "divine proportion" and featured it in many of his paintings, of which, most famously is the *Vitruvia Man*.

The golden ratio is not only reflected in science and art, but also business. In 2006, The Nobel Prize in Economics was awarded to Dr. Edmund Phelps, professor of economics at Columbia University. In announcing the prize, the Royal Swedish Academy of Sciences said Phelps's work had "deepened our understanding of the relation between short-run and long-run effects of economic policy." The golden ratio forms the mathematical foundation of Dr. Phelps' work in the Golden Rule of Capital Accumulation.

The golden ratio does exist. Then what is it? To answer the question, let's consider Euclid's definition of the golden ratio using the mathematical equation,

$$\frac{a+b}{a} = \frac{a}{b}.$$

Without loss of generality, we let $b = 1$ and replace the notation a with φ. The above equation turns to be

$$\frac{\varphi+1}{\varphi} = \frac{\varphi}{1}.$$

Clearing fractions and arranging terms, we have a quadratic equation

$$\varphi^2 - \varphi - 1 = 0.$$

From the quadratic formula, we obtain the golden ration φ exactly,

$$\varphi = \frac{1+\sqrt{5}}{2},$$

which is approximately equal to 1.61803. It is an irrational number. Just like the circumference ratio, there is a variety of activities students could get hands on experience with the golden ratio. Here are some examples. I ask my students to rewrite the golden ratio in different ways using patterns. For example, from the ratio equation, we can express in the recursive fashion,

$$\varphi = 1 + \frac{1}{\varphi}.$$

which gives a sequence representation of the golden ration,

$$\varphi = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}.$$

Meanwhile, the quadratics equation can also be arranged into another form,

$$\varphi^2 = 1 + \varphi.$$

Taking roots both sides, we obtain another sequence representation,

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}.$$

Both the circumference ratio and the golden ratio rise from geometry and they display the simplicity and beauty of nature. It's not necessary to have any advanced mathematical knowledge to understand them. Their deep interdisciplinary connection to the real world provides an effective learning environment for students. One project I assigned to students is to measure the rectangles for which they think are beautiful, so they have the firsthand experience with the golden ratio. The golden ratio can be observed in different ways from different objects including plants, animals and human bodies. It is also found from macroscale to microscale. In 2010, the journal Science reported the presence of the golden ratio at the atomic scale.

In my class, my goal is not only to help students develop mathematical thinking skills, but also to appreciate the interdisciplinary studies with mathematical foundation. To stimulate students' curiosity, I encourage students to investigate further applications in their own majors, such as business and biology.

3. The Euler's Number *e*

Compared to the long history of the circumference ratio π and the golden ratio φ, the Euler's number *e* is relatively young. It was named after the Swiss mathematician Leonard Euler, but it was discovered by another Swiss mathematician, Jacob Bernoulli in the calculation of compound interest in the seventeenth century.

Consider we have $100 ready to be deposited into a savings account. There are three bank offers to choose. The first bank offers a simple annual interest rate *r*, the second bank the same annual interest rate compounded quarterly, and the third one the same annual rate compounded monthly. For comparison, we compute the balance after one year with each bank.

With simple interest rate, the balance by the end of the 1st year is

$$100 \times (1 + r).$$

With interest compounded quarterly, the balance is

$$100 \times \left(1 + \frac{r}{4}\right)^4,$$

which is higher than that of the first offer.

Similarly, when compounded monthly, the balance is

$$100 \times \left(1 + \frac{r}{12}\right)^{12},$$

which is the highest among all three offers.

Moreover, following the patterns above, we generalize the calculation into a formula when the interest is compounded *n* times a year,

$$100 \times \left(1 + \frac{r}{n}\right)^n.$$

Now, we are ready for the case when the interest is compounded continuously. Essentially, the above formula requires us to calculate

$$\left(1 + \frac{1}{n}\right)^n$$

as → ∞ . It is a mathematical limit,

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n = e \approx 2.71828.$$

The limit is called the Euler's number, denoted by the letter *e*. Just like the circumference ratio and the golden ratio, the Euler's number is an irrational number. It is the base for both exponential functions and logarithmic functions.

Logarithmic functions are usually difficult to understand for undergraduates. In my class, one commonly asked question is why logarithms. To answer it, we continue to consider the above interest problem. We choose the bank which offers interests compounded continuously. We'd like to know how long it would take to reach certain monetary goals. For example, we'd like to know how long it would take to have the deposit doubled. As always, we use a notation to answer the question. Let us assume it will take *t* years. We need to solve an exponential equation,

$$2P = Pe^{rt},$$

which simplifies to,

$$2 = e^{rt}.$$

Here *P* is the principal and *r* is the interest rate. To solve for *t*, we need to create the new function, the natural logarithm *ln*,

$$t = \frac{1}{r}\ln 2.$$

The circumference ratio, the golden ratio and the Euler's number are all irrational numbers. Like the two ratios, the Euler's number can be represented by a convergent series,

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} \cdots.$$

Interestingly, for all three constants we studied, each can be represented by a series. The advantage of the series representation is that it only takes multiple arithmetic operations. The brain of a computer, the Central Processing Unit (CPU), can execute basic mathematical operations really fast. The series representation allows computer to handle complicated tasks including mathematical constants and transcendental functions efficiently. Mathematics is indispensable to technology.

Mathematical constants are building blocks of mathematics. They rise from real world applications which are important to our daily life. They cover a wide range of mathematical subjects from arithmetic, geometry to calculus. For the constants we discussed, they are connected to sequences and series and they may stimulate further interest in studying calculus.

Mathematics has a rich and diverse history and our knowledge of mathematics is still evolving. Mathematics education will have to evolve as we move forward. It takes generations of efforts to understand and accumulate mathematical knowledge. Meanwhile, it requires us, educators, to make changes so mathematics is more easily accessible to the new generation of students. I have been teaching mathematics and I am still learning because of the dynamics of mathematics. My next plan is to explore some fundamental mathematical theorems and their applications and deliver them effectively to students.

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