Motivating and Engaging Students in Active Learning of Mathematics Mathematical Theorems and Applications in Undergraduate Studies

Weiqun Zhang
Wright State Lake Campus

The purpose of this paper is to share with colleagues about the exploration in teaching and learning mathematics. Three most commonly used mathematical theorems and their applications in undergraduate studies are presented. The theorems are the Pythagorean theorem, the fundamental theorem of calculus, and the law of large numbers. The theorems are concrete mathematical materials to motivate and engage students with diverse backgrounds in an active learning environment. For each theorem, in addition to its brief history along with hands-on activities, critical thinking components with problem solving strategies are presented.

0. Introduction

Mathematics presents many challenges to both students and educators. It is often viewed as a tool for screening who should be allowed into majors including engineering, physics, chemistry, computer science and the biological sciences. More than 300,000 students are expected to the first semester calculus in universities and colleges across the United States. About a quarter of them will not earn a passing grade to continue. Many more are discouraged by their experience in learning mathematics, so they have to change their majors. To overcome the mathematics barrier for undergraduates, mathematics educators along with students are exploring a variety of new approaches to learn mathematics. The aim of an active learning environment is to promote active engagement with mathematics so that students not only understand mathematical concepts but also use their mathematical knowledge to solve real world problems.

As a longtime mathematics educator, the author has the privilege to work with many college students of diverse backgrounds. There are enthusiastic students who are really interested in mathematics and there are students who take mathematics simply as a course requirement for graduation. There are young students in their teens and adults in their middle career, or even after retirement. There are students with diverse culture background from different regions, states and countries. It’s also worth to note that there is a significant number of students, who are first generation college students. Each student is unique in his/her way of learning, preparedness and motive. As an educator, besides being an expert in the subject, one should be mindful to be flexible to adapt instruction materials and to present the mathematical content in accessible ways for individuals. The breadth and depth of fundamental mathematical theorems along with their applications allows mathematics educators to choose appropriate topics for students of diverse backgrounds.

Fundamental mathematical theorems occur at all levels in many undergraduate majors and they involve with many basic applications in human life. Here are a few examples in which mathematical theorems play important roles. The Pythagorean theorem was discovered in the very early stage of human civilizations. It provides concrete connection between geometry and algebra and it helped develop many branches of mathematical studies including trigonometric functions, mathematical analysis and number theory. The fundamental theorem of calculus is the pillar of modern mathematics. Its application ranges from areas and volumes, to fluid dynamics and rocket sciences. The law of large numbers describes the long term expected results at a large level. The application of the law of large numbers is not only reflected in natural science, but also business and social science.

In this paper, the author wishes to share with colleagues about the exploration in teaching and learning mathematics along with students. Specifically, three most important mathematical theorems and their applications in undergraduate studies are presented. These three theorems are the Pythagorean theorem, the fundamental theorem of calculus and the law of large numbers. The theorems are concrete mathematical materials to motivate and engage students with diverse backgrounds in an active learning environment. For each theorem, in addition to its brief history along with hands-on activities, critical thinking components with problem solving strategies are presented.

1. The Pythagorean Theorem

While mathematics is part of almost every aspect of everyday life, the Pythagorean Theorem is part of almost every field in undergraduate mathematical studies. The theorem shows that the fundamental relation in Euclid geometry among three sides of a right triangle. It states that "the area of the square built upon the hypotenuse of a right triangle is equal to the sum of the areas of squares upon the remaining sides." Algebraically, it is the mathematical equation

$$c^2 = a^2 + b^2$$

where *c* is the hypotenuse, and *a* and *b* are two short sides of a right triangle.

The theorem is named after the Greek mathematician and philosopher, Pythagoras (c. 570 – 495 BC). Although Pythagoras was credited with the theorem, there is evidence that Babylonians knew the result for some specific right triangles at lease one thousand years earlier. Indeed, the theorem was discovered independently by ancient civilizations including Mesopotamian, Indian and Chinese.

. Hands-on Activity

The Pythagorean Theorem demonstrates the simple connection between geometric concepts and algebraic representations. There are many hands-on activities which can be employed in classroom to improve and enhance the learning experience in algebra and geometry. Students are expected to understand multiple methods to prove the theorem. Geometrically, one can cut four congruent right triangles by arranging them in different positions to form squares. The theorem can be observed from areas. Algebraically, one can use the same ratio from similar triangles to derive the theorem. In fact, there are many more proofs for the Pythagorean theorem.

Another heuristic activity is to let students to find alternative ways to represent the theorem. For example, in the study of trigonometric functions, the theorem can be stated in the form of the Pythagorean identity,

$$sin^2\alpha + cos^2\alpha = 1,$$

where α is an angle. It can be further generalized into the Law of Cosines,

$$c^2 = a^2 + b^2 - 2ab\cos\gamma,$$

for any triangle with three sides *a*, *b* and *c*, and γ as the angle between *a* and *b*.

. Thinking Critically

One of the efficient and effective strategies in active learning is to challenge students with a question that uses the knowledge they have, but in an unfamiliar way. The Pythagorean theorem can serve as a classic example for this purpose. Since most students know the theorem well, the author presented another unusual proof using rate of change in calculus. The approach is not novel, but it does challenge students to apply what they learned previously in a new setting of classroom situation.

In the study of calculus, one can prove the theorem by considering the rate of change. If one short side *a* is increased by a small amount *da* by extending the side slightly to a fixed point, then the hypotenuse *c* is also increased by *dc*. Using the properties of similar triangles, we have a simple differential equation:

$$\frac{dc}{da} = \frac{a}{c}.$$

Applying the standard techniques of variable separation and integrating both sides, we obtain an integral equation:

$$\int cdc = \int ada.$$

It is easily solved with

$$c^2 = a^2 + K.$$

For the constant K from integration, we consider the extreme case when $a = 0$ and $c = b$. Therefore, we have the famous Pythagorean Theorem again,

$$c^2 = a^2 + b^2.$$

Another interesting aspect of the Pythagorean theorem is about the integral solutions of a more generalized algebraic equation

$$c^n = a^n + b^n,$$

for an integer *n*. When $n = 2$, it is exactly the Pythagorean Theorem and there are many integral solutions known as the Pythagorean triples. Here is a list of triples with values less than 100:

(3, 4, 5), (5, 12, 13), (7, 24, 25), (8, 15, 17), (9, 40, 41), (11, 60, 61), (12, 35, 37),

(13, 84, 85), (16, 63, 65), (20, 21, 29), (28, 45, 53), (33, 56, 65), (36, 77, 85),

(39, 80, 89), (65, 72, 97)

When $n > 2$, the French mathematician Pierre de Fermat conjectured that there were no integral solutions. His conjecture was proved by Andrew Wiles of Princeton University in 1993 and it's now known as the Fermat's Last Theorem. Hundred years of effort in proving Fermat's conjecture led the modern development of two new branches of mathematics, functional analysis and number theory. The number theory is the foundation to the development of modern cryptography in computer science.

. Problem Solving

The Pythagorean theorem is a good subject for interdisciplinary studies. Students can use skills, tools and theories in mathematics to enhance the learning experience in computer science. For example, in my computer programming class, the author assigned students to write a program to generate the Pythagorean triples. Let *m* and *n* be integers and $m > n$. Using the formula given in Euclid's Element, we obtain the Pythagorean triple:

$$a = 2mn,$$
$$b = m^2 - n^2, \text{ and}$$
$$c = m^2 + n^2.$$

Using simple and concrete mathematical concepts, students are able to design and implement computer software, which is a very important concept for students to learn and to comprehend. It is accomplished through the use of technology, manipulatives, and analytical skills.

2. The Fundamental Theorem of Calculus

The fundamental theorem of calculus connects two building blocks of calculus, derivates and integrals. The theorem shows derivatives and integrals are essentially mutual inverses. For the purpose of this paper, we only consider the part of the fundamental theorem of calculus with more applications in undergraduate studies. It states that for a continuous function $f(x)$ on a closed interval $[a, b]$,

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$, or $F'(x) = f(x)$.

The development the fundamental theorem of calculus can be traced back more than two thousand years ago. One of the greatest mathematicians, Archimedes (c. 287 – c. 212 BC) opened the door for modern calculus by applying the idea of infinitesimals and the method of exhaustion. He was able to derive rigorously the area of the circle, the surface area and the volume of a sphere. In the early seventeenth century, Newton (1642 – 1727) completed the development the mathematical theory and formalized it.

. Hands-on Activity

The integral of a curve can be thought as the area of a curve under the graph and above the X-axis between two defined boundaries. The area under the graph is breaking into infinitesimally thin columns. By definition, an integral is a Riemann sum of similar products, each of which represent the small area of a thin column. The Riemann sum approach offers a visual way to see the total area as a sum of small areas. Besides it makes sense of other contexts that students will see as they develop concepts and skills in applications of integrals. For example, in working with volumes of revolution, using differential notation such as $dV = \pi r^2 dx$, reminds students that an integral is an infinite sum of products of a function and an infinitesimal change dx along the X-axis. Specifically, the differential dx is an active factor of a product and it is essential to include it in the integral notation. Clear discussion of the fundamental theorem of calculus in combination with this development via differentials will also clarify and solve another an issue when to consider integrating with respect to *x* or *y*. When presenting the theorem in class, the author uses introductory examples of areas to let students develop a strong sense of the meaning from visual representations of thin columns either horizontally or vertically. A good understanding of visual concepts for areas as Riemann sums, can be broadened to find volumes, arc lengths, work done by a variational force. For an educator, it is crucial to help students not only develop skills in evaluation of definite integrals using the fundamental theorem of calculus, but also visualize the meaning of integrals.

. Thinking Critically

According to criticalthinking.org, critical thinking is that mode of thinking – about any subject, content, or platform – in which the thinker improves the quality of one's thinking by skillfully analyzing, assessing, and restructuring it.

The fundamental theorem of calculus is a milestone of calculus. It is one of the most challenging topics in undergraduate studies. Moreover, with improved thinking skills in analyzing and assessing, the fundamental theorems of calculus can be restructured in other forms. In vector calculus, a force field is represented by a vector field $\vec{F}$and a curve by a vector valued function $\vec{r}$. The work done by a force on an object moving along the curve can be found as the line integral $\int_C \vec{F} \cdot d\vec{r}$, which in most cases requires a significant amount of computing effort. However, when $\vec{F}$ is conservative, like the fundamental theorem of calculus, there is the fundamental theorem of line integrals,

$$\int_C \mathrm{F} \cdot d\vec{r} = f\left(\vec{r}(b)\right) - f\left(\vec{r}(a)\right),$$

where *C* is a smooth curve given by the vector function $\vec{r}(t)$ for $a \le t \le b$ and $\vec{F}$ is the gradient of a scalar valued function f, $\vec{F} = \nabla f$. In other words, both theorems can be stated in a similar fashion. That is, to compute the integral of a derivative, it is only necessary to consider the change of the original function values at the endpoints.

From physics, we know that the earth's gravity field $\vec{F}$ is conservative,

$$\vec{F}(x, y, z) = -\frac{mMG}{\left(\sqrt{x^2+y^2+z^2}\right)^3} < x, y, z >,$$

where *M* is the mass of the earth and *m* is the mass of an object at any location (x, y, z). We also know that $\vec{F} = \nabla f$, where $f(x, y, z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$. We calculate the work done by the gravity on a moving object of mass *m* from the point (5, 4, 1) to another point (2, 0, 3) along a smooth curve C. By the fundamental theorem of line integrals, the work done is

$$W = \int_C \mathrm{F} \cdot d\vec{r} = f\left(\vec{r}(b)\right) - f\left(\vec{r}(a)\right)$$

$$f(2, 0, 3) - f(5, 4, 1) = mMG\left(\frac{1}{\sqrt{13}} - \frac{1}{\sqrt{42}}\right).$$

The approach to evaluate the line integral is very similar to that of the fundamental theorem of calculus and it is much more efficient compared to the direct calculation by definition. More importantly, it shows that the line integral is path independent if $\vec{F}$ is conservative. The mathematical observance of the path independence matches exactly the energy conservation principle in physics.

While the fundamental theorem of line integrals answers one aspect of a line integral when $\vec{F}$ is conservative, the Green's theorem answers another aspect of a line integral when the curve involved is closed. It states that for a two-dimensional vector field $\vec{F} =< P, Q >$ such that *P* and *Q* have continuous partial derivatives on an open region containing *D*,

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA,$$

where *D* is the region bounded by a smooth, simply closed and positively oriented curve *C*. In short, the Green's theorem simply says a line integral on a closed curve can be evaluated as a double integral over the region enclosed.

Here is another good example to challenge students' critical thinking skill on a familiar topic but under a different setting. Let's consider to the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Using polar coordinates, we get the ellipse's parametric equations,

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases} \text{ for } 0 \le t \le 2\pi.$$

Let *A* denote the area,

$$A = \iint_D 1dA = \frac{1}{2}\oint xdy - ydx$$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos t)\,(b\cos t)dt - (b\sin t)(-b\cos t)dt.$$

From the Pythagorean identity $cos^2t + sin^2t = 1$, we have the area of the region enclosed by the ellipse $A = \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi ab$.

. Problem Solving

We consider a basic engineering problem: A 200-lb cable is 100 ft long and hangs vertically from top of a tall building. How much work is required to lift the cable to the top of the building? Let's place the origin at the top of the building and the X-axis pointing downward. We divide the cable into small parts with length Δx. If x_i^* is a point in the *i*th such interval, then all in the interval are lifted approximately the same amount, namely x_i^*. The cable weighs 2 pounds per foot, so the weight of the *i*th part is $2\Delta x$. Then the work done on the *i*th part, is

$$(2\Delta x)\cdot x_i^* = 2x_i^*\Delta x.$$

We get the total work done by adding all these approximations. By letting the number of parts become large (so $\Delta x \to 0$), the work done is a Riemann sum, i.e., an integral:

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} 2x_i^*\Delta x = \int_0^{100} 2xdx = x^2|_0^{100} = 10{,}000\ lb\cdot ft.$$

3. The Law of Large Numbers

In probability and statistics, the law of large numbers states that the sample mean approaches to the population mean as the sample size increases. The first proof of the theorem was derived by Jacob Bernoulli for a binomial random variable in the early eighteenth century.

. Hands-on Activity

The discovery of the theorem demonstrates the power of innovative and creative thinking from intuition and common sense. Students can learn the theorem using hands-on activities such as flipping coins and rolling dice. In both experiments, students are required to record the experimented results using charts and they are required to repeat the experiments for a relatively large number of trials. The goal of the experiments is to enhance innovative and creative thinking abilities of undergraduate students. Furthermore, students are expected to take advantages of the skills they learned in the activities to explore their own field of studies in science, engineering and technology.

. Thinking Critically

One important aspect of critical thinking on the law of large numbers, is its impact on growth rates, such as in population growth and business growth. It is common sense that high growth rates are not sustainable. In the business context, the law of large numbers dictates the growth rates of businesses. Accordingly, it simply indicates that as a company grows, it is more and more difficult to sustain its previous growth rates, which means the growth rate may decline while the company grows.

. Problem Solving

Another big application of the law of large numbers is in the insurance industry. While there is no guaranty that an insurance company will make money on each individual policyholder, it is expected that the insurance company will make money when there are enough policy holders. The law of large numbers is used to calculate the premium for the insurance policy. Suppose that an insurance company sells an employment security policy that pays you $200,000 in the event that you quit your job because of serious illness. Based on data from past claims, the probability that a policyholder will make a claim for loss of a job is 1 in 1000. Should the insurance company expect to earn a profit if it sells the policies for $250 each?

Clearly, if the insurance company sells only a few policies, there is no certainty on profit or loss. In an extreme case, if it sells 100 policies, and none of 100 policyholders files a claim, the insurance company will take the entire revenue $25,000 as a profit. But if even one policyholder makes a $200,000 claim. The insurance faces a net loss of $175,000.

However, if it sells a large number of policies, the law of large numbers tells us that the proportion of policies for which claims must be paid should be very close to the 1 in 1000 probability for the policy. For example, if it sells 1 million policies, the insurance company should expect that the number of policyholders making the $200,000 claim will be very close to

$$1{,}000{,}000 \times \frac{1}{1{,}000} = 1000.$$

Paying these 1000 claims will cost

$$1000 \times \$200{,}000 = \$200\ million.$$

Therefore, the average cost of each of the one million policyholders is $200. When the policy is sold at $250 each, the insurance company is expected to earn the amount of $50 million as profit for one million policies sold.

Teaching mathematics is a complex practice demanding both creative and critical thinking from students as well as educators. In planning and conducting instruction, mathematics educators make sure that students understand that abstract mathematical ideas and theorems are derived to solve real world problems. The three mathematical theorems presented are building blocks of mathematics and they cover a full range of mathematical subjects from arithmetic, geometry and algebra to calculus and probability theory and statistical analysis. Besides its rich history and broad application, mathematics is still evolving. Mathematics education will have to evolve to accommodate new generations of students. An active mathematics learning environment is composed of concrete learning material, active student participation and collaboration with each other. A big goal in teaching mathematics is to use mathematical theorems to demonstrate a process how critical and creative thinking leads to the discovery of new knowledge. Meanwhile, students are able to analyze mathematical arguments with hands-on activities and apply mathematical concepts to solve real world problems.

References

Bennett and Briggs *Using & Understanding Mathematics – A Quantitative Reasoning Approach*, 5th Edition, Addison Wesley.

Blair, Kirkman, and Maxwell *Statistical Abstract of Undergraduate Programs in the Mathematical Sciences in the United States.* Retrieved from http://www.ams.org/profession/data/cbms-survey/cbms2015

Cooke, R.L. *The History of Mathematics: A Brief Course*, Wiley & Sons, 2013.

The Mathematical Sciences in 2025, The National Academies Press, www.nap.edu

Protheroe, N. *What Does Good Math Instruction Look Like?* 2007. Retrieved from https://www.naesp.org/sites/default/files/resources/2/Principal/2007/S-Op51.pdf.

The Royal Swedish Academy of Sciences *Edmund Phelps's Contributions to Macroeconomics*, 2006. Retrieved from https://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/2006/advanced-economicsciences2006.pdf.

Stewart, J. *Calculus – Concepts & Contexts*, 4e, Brooks/Cole Cengage Learning, 2010.